Mathematics

1. A value of θ for which $\dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ purely imaginary is

(1) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ **(2)** $\dfrac{\pi}{3}$

(3) $\dfrac{\pi}{6}$ **(4)** $\sin^{-1}\left(\dfrac{\sqrt{3}}{4}\right)$

Sol. [1] $\dfrac{2+3i\sin\theta}{1-2i\sin\theta}=\dfrac{(2+3i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta}$

$=\dfrac{\left(2-6\sin^2\theta\right)+i7\sin\theta}{1+4\sin^2\theta}$

$\Rightarrow 2-6\sin^2\theta=0 \qquad \Rightarrow \sin\theta=\dfrac{1}{\sqrt{3}}.$

2. The system of linear equations

$x+\lambda y-z=0$

$\lambda x-y-z=0$

$x+y-\lambda z=0$

has a non-trivial solution for

(1) exactly three values of λ

(2) infinitely many values of λ

(3) exactly one value of λ

(4) exactly two values of λ

Sol. [1] $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix}=0 \quad \Rightarrow (\lambda+1)\lambda(\lambda-1)=0$

$\Rightarrow \lambda=-1, 0, 1.$

3. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then

(1) $2x=r$ **(2)** $2x=(\pi+4)r$

(3) $(4-\pi)x=\pi r$ **(4)** $x=2r$

Sol. [4] Let length of pieces be ℓ, $2-\ell$

$\therefore\ 4x=\ell,\ 2\pi r=2-\ell \ \Rightarrow\ x=\dfrac{\ell}{4},\ r=\dfrac{2-\ell}{2\pi}$

Area $=A=x^2+\pi r^2=\dfrac{\ell^2}{16}+\dfrac{\pi}{4\pi^2}(2-\ell)^2$

$A'(\ell)=\dfrac{2\ell}{16}-\dfrac{2}{4\pi}(2-\ell)=0$

$\Rightarrow\ \ell=\dfrac{8}{\pi+4}$

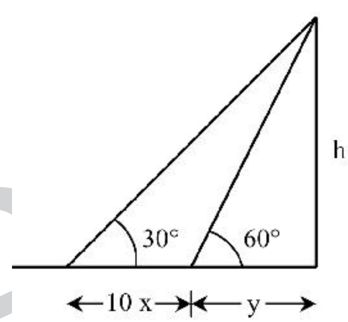
$\therefore\ \dfrac{x}{r}=\dfrac{\ell\pi}{2(2-\ell)}=\dfrac{\dfrac{8}{\pi+4}\pi}{2\left(2-\dfrac{8}{\pi+4}\right)}=2.$

4. A man is walking towards a vertical pillar in a straight path, at a unitform speed. At a certain point A on the path, he observed that the angle of elevation ofthe top of the pillar is 30°. AFter walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, from B to reach the pillar, is

(1) 5 **(2)** 6

(3) 10 **(4)** 20

Sol. [1] Let speed be x m/ min.



$\dfrac{h}{y}=\tan 60°=\sqrt{3} \ \Rightarrow\ h=\sqrt{3}y$

and $\dfrac{h}{10x+y}=\tan 30°=\dfrac{1}{\sqrt{3}}$

$\Rightarrow\ 3y=10x+y$

$\Rightarrow\ y=5x$

$\therefore$ time $=5$ min.

5. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is **NOT true**?

(1) E_1, E_2 and E_3 are indepedent

(2) E_1 and E_2 are independent

(3) E_2 and E_3 are indepedent

(4) E_1 and E_3 are indepedent

Sol. [1] $P(E_1)=\frac{1}{6},\ P(E_2)=\frac{1}{6},\ P(E_3)=\frac{18}{36}=\frac{1}{2}$

$P(E_1\cap E_2)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

$P(E_2\cap E_3)=\frac{3}{36}=\frac{1}{12}=P(E_1\cap E_3).$

6. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?

(1) $3a^2-23a+44=0$

(2) $3a^2-26a+55=0$

(3) $3a^2-32a+84=0$

(4) $3a^2-34a+91=0$

Sol. [3] $\sigma^2=\frac{\sum xi^2}{n}-\left(\frac{\sum xi}{n}\right)^2=(3.5)^2$

$\Rightarrow\ \frac{a^2+134}{5}-\left(\frac{a+16}{5}\right)^2=12.25$

$\Rightarrow\ 3a^2-32a+84=0.$

7. For $x\in\mathbf{R}$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then

(1) g is differentiable at x = 0 and $g'(0)=-\sin(\log 2)$

(2) g is not differentiable at x = 0

(3) $g'(0)=\cos(\log 2)$

(4) $g'(0)=-\cos(\log 2)$

Sol. [3] $g(x)=\left|\ln 2-\sin|\ln 2-\sin x|\right|$ in nbd of 0

$g(x)=\ln 2-\sin(\ln 2-\sin x)$

$\therefore\ g'(x)=-\cos(\ln 2-\sin x)(-\cos x)$

$\therefore\ g'(0)=\cos(\ln 2).$

8. The distance of the point $(1,-5,9)$ from the plane $x-y+z=5$ measured along the line $x=y=z$ is

(1) $\frac{20}{3}$ **(2)** $3\sqrt{10}$

(3) $10\sqrt{3}$ **(4)** $\frac{10}{\sqrt{3}}$

Sol. [3] Line through $P(1,-5,9)$ parallel to given line

$\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1}=r$ ...(1)

$\Rightarrow$ Point of int. of (1) with plane be

$Q(r+1,\ r-5,\ r+9)$

lies on the plane

$\therefore\ (r+1)-(r-5)+(r+9)=5\ \Rightarrow\ r=10$

$\therefore\ Q(-9,-15,-1)\ \Rightarrow\ PQ=10\sqrt{3}.$

9. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

(1) $\sqrt{3}$ **(2)** $\frac{4}{3}$

(3) $\frac{4}{\sqrt{3}}$ **(4)** $\frac{2}{\sqrt{3}}$

Sol. [4] $2b=c$

$\Rightarrow\ 4b^2=c^2=a^2+b^2\ \Rightarrow\ \frac{b^2}{a^2}=\frac{1}{3}$

$\therefore\ e=\sqrt{1+\frac{b^2}{a^2}}=\frac{2}{\sqrt{3}}.$

10. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y+6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$

(2) $x^2 + y^2 - 4x + 8y + 12 = 0$

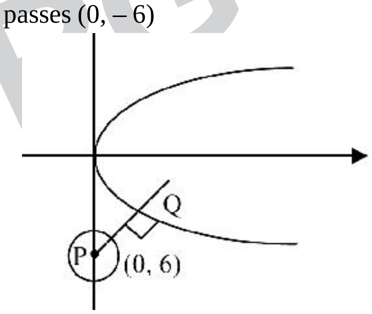
(3) $x^2 + y^2 - x + 4y - 12 = 0$

(4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

Sol. [2] Let normal through P be

$y = mx - 4m - 2m^3$ ...(1)

it passes (0, – 6)



$\Rightarrow\ -6 = 0 - 4m - 2m^3 \quad \Rightarrow\ m = 1$

$\therefore\ Q = (2m^2, -4m) = (2, -4)$

$\therefore$ Circle $(x-2)^2 + (y+4)^2 = 8$

$\Rightarrow\ x^2 + y^2 - 4x + 8y + 12 = 0.$

11. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\,\text{adj}A = A A^T$, then 5a + b is equal to

(1) 13 **(2)** –1

(3) 5 **(4)** 4

Sol. [3] $A \text{ adj } A = AA^T \quad \Rightarrow |A|I = AA^T$

$\Rightarrow\ (10a + 3b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$15a - 2b = 0$

$10a + 3b = 13$

$\therefore\ a = \frac{2}{5},\ b = 3$

$5a + b = 5.$

12. Consider

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right).$$

A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the point

(1) $\left(\frac{\pi}{4}, 0\right)$ **(2)** $(0, 0)$

(3) $\left(0, \frac{2\pi}{3}\right)$ **(4)** $\left(\frac{\pi}{6}, 0\right)$

Sol. [3] $y = f(x) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}$

when $x = \frac{\pi}{6},\ y = \frac{\pi}{3}$ and $\frac{dy}{dx} = \frac{1}{2}$

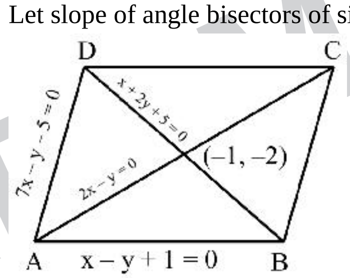
Normal $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right) \quad \Rightarrow\ 2x + y = \frac{2\pi}{3}.$

13. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at (–1, –2), then which one of the following is a vertex of this rhombus?

(1) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ **(2)** $(-3, -9)$

(3) $(-3, -8)$ **(4)** $\left(\frac{1}{3}, -\frac{8}{3}\right)$

Sol. [4] Let slope of angle bisectors of sides be m



$\therefore\ \frac{2m}{1-m^2} = \frac{7+1}{1-7} \quad \Rightarrow\ m = 2, -\frac{1}{2}$

Diagonals $(y+2) = 2(x+1),\ (y+2) = -\frac{1}{2}(x+1)$

$\Rightarrow\ 2x - y = 0,\ x + 2y + 5 = 0$

$\therefore\ A(1, 2),\ B\left(\frac{7}{3}, -\frac{4}{3}\right),\ D\left(\frac{1}{3}, -\frac{8}{3}\right).$

14. If a curve $y = f(x)$ passes through the point $(1,-1)$ and satisfies the differential equation, $y(1+xy)dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(1) $\frac{4}{5}$ **(2)** $-\frac{2}{5}$

(3) $-\frac{4}{5}$ **(4)** $\frac{2}{5}$

Sol. [1] $ydx - xdy + xy^2dx = 0$

$\Rightarrow \frac{ydx - xdy}{y^2} + xdx = 0 \quad \Rightarrow d\left(\frac{x}{y}\right) + d\left(\frac{x^2}{2}\right) = 0$

$\Rightarrow \frac{x}{y} + \frac{x^2}{2} = C$ ...(1)

it pases (1, –1) $\Rightarrow -1 + \frac{1}{2} = C$

$\Rightarrow C = -\frac{1}{2}$

$\therefore \frac{x}{y} + \frac{x^2}{2} = -\frac{1}{2} \quad \Rightarrow \frac{x}{y} = -\frac{(x^2+1)}{2}$

$\Rightarrow f(x) = y = -\frac{2x}{x^2+1}$

$\therefore f\left(-\frac{1}{2}\right) = \frac{5}{4}.$

15. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is

(1) 58[th] **(2)** 46[th]

(3) 59[th] **(4)** 52[nd]

Sol. [1] Starting with $A = \frac{4!}{2!} = 12$

Starting with $L = 4! = 24$

Starting with $M = \frac{2!}{2!} = 12$

Starting $SA = \frac{3!}{2!} = 3$

Starting with $SL = 3! = 6$
Next : SMALL
Rank = 58.

16. If the 2[nd], 5[th] and 9[th] terms of a non-constant A.P. are in G.P., then the commonr atio of this G.P. is

(1) $\frac{7}{4}$ **(2)** $\frac{8}{5}$

(3) $\frac{4}{3}$ **(4)** 1

Sol. [3] $a+d,\ a+4d,\ a+8d$ in G.P.

$\Rightarrow (a+4d)^2 = (a+d)(a+8d)$

$\Rightarrow 8d^2 = ad \qquad \Rightarrow d = \frac{a}{8}$

Now $r = \frac{a+4d}{a+d} = \frac{a+\frac{a}{2}}{a+\frac{a}{8}} = \frac{4}{3}.$

17. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n, x \neq 0$, is 28, then the sum of the coefficients fo all the terms in this expansion, is

(1) 729 **(2)** 64

(3) 2187 **(4)** 243

Sol. [1] Number of term = ${}^{n+r-1}C_{r-1}$

$= {}^{n+2}C_2$

Now ${}^{n+2}C_2 = 28 \quad \Rightarrow n = 6$

sum of coefficient $= \left(1 - \frac{2}{1} + \frac{4}{1}\right)^6 = 3^6.$

18. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$, is $\frac{16}{5}m$, then m is equal to

(1) 99 **(2)** 102

(3) 101 **(4)** 100

Sol. [3] $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$

$\frac{(8^2 + 12^2 + 16^2 + 20^2 + \ldots + 10\text{terms})}{5^2} = \frac{16}{5}m$

$\Rightarrow 2^2+3^2+4^2+5^2+\ldots=5m$

$\dfrac{11\times12\times23}{6}-1=5m \quad \Rightarrow m=101.$

19. If the line, $\dfrac{x-3}{2}=\dfrac{y+2}{-1}=\dfrac{z+4}{3}$ lies in the plane, $\ell x+my-z=9$, then ℓ^2+m^2 is equal to

(1) 2 **(2)** 26
(3) 18 **(4)** 5

Sol. [1] $2\ell+m(-1)-1(3)=0 \quad \Rightarrow 2\ell-m=3$

and $3(\ell)+2(-2)(m)+4=9$

$\Rightarrow 3\ell-2m=5 \quad \Rightarrow \ell=1,\ m=-1.$

20. The Boolean Expression $(p\wedge\sim q)\vee q\vee(\sim p\wedge q)$ is equivalent to

(1) $p\vee\sim q$ **(2)** $\sim p\wedge q$
(3) $p\wedge q$ **(4)** $p\vee q$

Sol. [4]

p	q	~p	~q	(p∧~q)	(~p∧q)	(p∧~q)∨q∨(~p∧q)	(p∨q)
T	T	F	F	F	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	F	F	F

21. The integral $\displaystyle\int\frac{2x^{12}+5x^9}{\left(x^5+x^3+1\right)^3}dx$ is equal to

(1) $\dfrac{-x^{10}}{2\left(x^5+x^3+1\right)^2}+C$ **(2)** $\dfrac{-x^5}{\left(x^5+x^3+1\right)^2}+C$

(3) $\dfrac{x^{10}}{2\left(x^5+x^3+1\right)}+C$ **(4)** $\dfrac{x^5}{2\left(x^5+x^3+1\right)^2}+C$

where C is an arbitrary constant.

Sol. [3] $\displaystyle\int\frac{2x^{12}+5x^9}{\left(x^5+x^3+1\right)^3}dx=\int\frac{2x^{-3}+5x^{-6}}{\left(1+x^{-2}+x^{-5}\right)^3}dx$

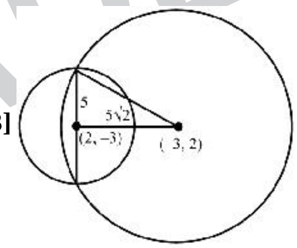
$\displaystyle=\int-\frac{dt}{t^3}$ let $1+x^{-2}+x^{-5}=t$

$=\dfrac{t^2}{2}+C=\dfrac{x^{10}}{2\left(x^5+x^3+1\right)}+C.$

22. If one of the diameters of the circle, given by the equation, $x^2+y^2-4x+6y-12=0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is

(1) 10 **(2)** $5\sqrt{2}$
(3) $5\sqrt{3}$ **(4)** 5

Sol. [3]



$r^n=25+50 \quad \Rightarrow r=5\sqrt{3}.$

23. $\displaystyle\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots3n}{n^{2n}}\right)^{1/n}$ is equal to

(1) $3\log3-2$ **(2)** $\dfrac{18}{e^4}$

(3) $\dfrac{27}{e^2}$ **(4)** $\dfrac{9}{e^2}$

Sol. [3] $\displaystyle L=\lim_{n\to\infty}\left[\frac{(n+1)(n+2)\ldots(n+2n)}{n^{2n}}\right]^{1/2}$

$\displaystyle\ln L=\lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+2}{n}\right)+\ldots+\ln\left(\frac{n+2n}{n}\right)\right]$

$\displaystyle=\int_0^2\ln(1+x)dx$

$\Rightarrow \ln L=3\ln3-2 \quad \Rightarrow L=\dfrac{27}{e^2}.$

24. The centre of those circles which touch the circle, $x^2+y^2-8x-8y-4=0$, externally and also touch the x-axis, lie on

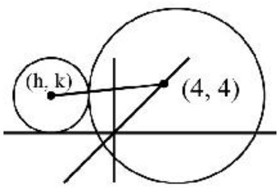
(1) a parabola

(2) a circle

(3) an ellipse which is not a circle

(4) a hyperbola

Sol. [1]



$$(h-4)^2+(k-4)^2=|k|+4.$$

25. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{5\pi}{6}$ **(2)** $\frac{3\pi}{4}$

(3) $\frac{\pi}{2}$ **(4)** $\frac{2\pi}{3}$

Sol. [1] $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$$

$$\Rightarrow \vec{a}\cdot\vec{b}=-\frac{\sqrt{3}}{2} \qquad \Rightarrow \cos\theta=-\frac{\sqrt{3}}{2}.$$

26. Let $p=\lim_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{1/2x}$ then log p is equal to

(1) $\frac{1}{4}$ **(2)** 2

(3) 1 **(4)** $\frac{1}{2}$

Sol. [4] $P=\lim_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{1/2x}$

$$P=e^{\lim_{x\to 0+}\frac{\tan^2\sqrt{x}}{2x}}$$

$$P=e^{1/2}.$$

27. If $0\le x<2\pi$, then the number of real values of x, which satisfy the equation

$\cos x+\cos 2x+\cos 3x+\cos 4x=0$ is

(1) 9 **(2)** 3

(3) 5 **(4)** 7

Sol. [4] $\cos x+\cos 2x+\cos 3x+\cos 4x=0$

$$\Rightarrow 2\cos x\cos 2x+2\cos x\cos 3x=0$$

$$\Rightarrow 2\cos x\cdot\cos\frac{5x}{2}\cdot\cos\frac{x}{2}=0$$

$$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2},\pi,\frac{\pi}{5},\frac{3\pi}{5},\frac{7\pi}{5},\frac{9\pi}{5}.$$

28. The sum of all real values of x satisfying the equation

$\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is

(1) 5 **(2)** 3

(3) –4 **(4)** 6

Sol. [2] $\left(x^2-5x+5\right)^{x^2+4x-60}=1$

$x^2+4x-60=0 \quad \Rightarrow (x+10)(x-6)=0$

$\Rightarrow x=-10, 6$

$x^2-5x+5=1 \quad \Rightarrow (x-4)(x-1)=0$

$\Rightarrow x=1, 4$

$x^2-5x+5=-1 \quad \Rightarrow x^2-5x+6=0$

$\Rightarrow (x-2)(x-3)=0$

$\Rightarrow x=2, 3$ (3 ↓ rejected)

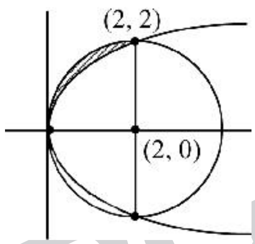
Solution set = $\{-10, 6, 1, 4, 2\}$.

29. The area (in sq. units) of the region $\{(x,y): y^2 \geq 2x \text{ and } x^2+y^2 \leq 4x,\ x \geq 0,\ y \geq 0\}$ is

(1) $\frac{\pi}{2}-\frac{2\sqrt{3}}{3}$ **(2)** $\pi-\frac{4}{3}$

(3) $\pi-\frac{8}{3}$ **(4)** $\pi-\frac{4\sqrt{2}}{3}$

Sol. [3]



Req. Area : $2\pi-\int_0^2 \sqrt{2x}\,dx-\pi$

$=\pi-\frac{\sqrt{2}\cdot 2\sqrt{2}}{\left(\frac{3}{2}\right)}=\pi-\frac{8}{3}.$

30. If $f(x)+2f\left(\frac{1}{x}\right)=3x,\ x \neq 0$, and

$S=\{x \in \mathbf{R} : f(x)=f(-x)\}$; then S

(1) contains more than two elements

(2) is an empty set

(3) contains exactly one element

(4) contains exatly two elements

Sol. [4] $f(x)+2f\left(\frac{1}{x}\right)=3x$ ...(1)

$f\left(\frac{1}{x}\right)+2f(x)=\frac{3}{x}$

$\therefore\ f(x)=\frac{2}{x}-x$

Now $f(x)=f(-x)$ $\Rightarrow \frac{2}{x}-x=\frac{-2}{x}+x$

$\Rightarrow x^2=\frac{1}{2}$ $\Rightarrow x=\pm\frac{1}{\sqrt{2}}.$